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THE SOUND OF MATHEMATICS

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(1) Introduction

Visual representations of mathematical objects can produce intriguing and beautiful patterns. A recent example is given by the work of Benoit MANDELBROT on fractals and the research group "Komplexe Dynamik" at the University of Bremen ("JULIA sets"). In our lecture we address the problem of acoustical representations of mathematical objects. We will listen to some of the sounds produced by mathematical objects.

(2) Variations on prime numbers

The medium of music is time. Time is structured by the creation of rhythm. Mathematically, a rhythm is given by a strictly monotone sequence of natural numbers: $a_1, a_2, \ldots, a_n, \ldots,$ Here a_n specifies the attack time of the nth tone in a composition and a. can be chosen as a natural number once a basic uniform quantization of time is agreed upon.

Uniform rhythms are specified by sequences whose first order differences are periodic:

Whose rhythm is generated by the sequence of prime numbers:

This is certainly not a completely regular rhythm, but it reveals a remarkable amount of structure and fascinating patterns if it is compared to a completely random rhythm.

The rhythm is enforced if further percussive voices with regular beats are added, for instance

This structure illustrates the sieve of ERATHOSTENES, where all multiples of the prime numbers already found are weeded out in the natural number sequence. Several other variations have been developed:

Prime number alternating walk with antiparallel voices, Tonal phrases between maximal prime number distances etc.

(3) The rhythm of the numbers

An alternative and completely isomorphous description of a rhythmic structure is via the specification of an infinite 0-1-sequence $s_1, s_2, \ldots, s_n, \ldots$, where $s_n = 1$ if time point number n is attacked.

The 0-1-sequence corresponds to the prime number sequence

0110101000101000101000101000…

Several other mathematical objects are most naturally given as infinite 0-1-sequences.

The set of all such sequences can be made into a topological space (the so-called shift space) by imposing the shift topology upon it. In this space several "nearly periodic" sequences have been studied which produce highly regular but nonperiodic rhythms. Some examples are possible: Rhythms in shift space, The MORSE sequence, An infinite waltz.

A simple way to create many interesting 0-1-sequences is to study the binary expansion of real numbers. Rational fractions yield uniform rhythms:

Aperiodic rhythms appear when one expands irrational numbers, like for example:

11.0010010000111111011010100010001000010000101…

(4) The sound of chaos

The compositions discussed above could be played on any MIDI-controllable instrument. For the following experiments, however, somewhat more elaborate equipment is required, since sounds are directly computed by a digital computer. We used an IBM/AT together with an individually manufactured linear 16-bit high resolution D/A-converter which is controlled by an STSC/APL environment with some embedded assembler functions.

It is well known that the differential equation of the undamped unforced linear oscillator:

$V^2x(t) + (2\ f)^2x(t) = 0$

has as solution a pure sinus wave of normalized frequency f

$x(t) = a\ \sin(2ft\text{-ø})$,

where amplitude a and phase ø are determined by the initial conditions x(0) and Vx(0). By variation of the basic differential equation, more complex wave forms can be obtained. A simple example is given by the undamped unforced nonlinear oscillator

$V^2x(t) + (2\ f)^2\sin(x(t)) = 0$

which produces non-sinusoidal waveforms with constant amplitude and period and spectral power at frequencies w, 3w, 5w,…, where the frequency w depends on the given initial conditions.

In recent years many scientists have studied the phenomenon of chaotic motions in such differential equations. The non-dimensionalized equation of an impact oscillator can be written as

$V^2x(t) + (2z\ /h\ )D(x(t)) = (1/4h^2)\sin(2\ ft)$

This is a forced oscillator which generates steady state chaotic waveforms with approximate subharmonics, depending on the parameters h and z . Steady state trajectories are depicted in THOMSON and STEWART (1986), p. 318, where the behaviour of the system is studied in detail.

In general, every oscillator can be investigated by obtaining the trajectory generated by the system from given initial conditions via some stable numerical integration routine. This gives a sequence of real numbers, which can be converted to an audio signal with the help of a D/A converter.

Chaotic solutions tend to produce strange sounds with some noise-like components or sounds which shift between different waveforms in an irregular and unpredictable fashion. Fascinating effects can be obtained by slow parameter variations which move the system from one attractor basin to another.

The use of such dynamic systems for sound synthesis, fascinating as it might be, does not lead to predictable results. We have, however, created an innovative approach to sound generation which is based on a different notion from the theory of fractals. The method is called additive fractal synthesis. The fundamental idea behind this method can be understood if one looks at a tree. At least approximately, such a tree is self-similar in the sense, that the whole of the tree can be mapped by an affine contracting transformation to its parts. Exactly the same approach can be used for the synthesis of sounds. The examples to be presented at the conference will be sounds that have, in our opinion, a liveliness comparable to the sound of acoustical instruments, but possess completely controllable looping points.

(5) The emergence of regular structures

Consider a liquid metal at high temperatures. If the metal is slowly cooled, thermal mobility is gradually lost and it is possible that the atoms are able to align themselves in a perfect crystal that is completely ordered over a distance up to billions of times the size of an individual atom. There exists an algorithm by METROPOLIS et al. (1953) and others (see f.i. PRESS et

al. (1986)), which simulates this slow cooling process in the optimization of combinatorial structures.

A composition can be regarded as a combinatorial structure that possesses a certain energy for instance defined as the number of violations of the rules given by the strict theory of polyphonic counterpoint. This energy can gradually be lowered with the goal to produce a crystalline structure with given properties.

Several examples can be presented: Four voices chasing one another, The emergence of counterpoint from noise.

(6) An open mathematical problem

A paper on mathematics would not be complete without an unsolved mathematical problem. Legend has it that the solution was in the possession of the pythagoreans. We, certainly, do not know a proof today.

Consider the sequence of all those numbers which have merely the first three prime numbers 2, 3, 5 in their prime factorization. The sequence begins as follows:

1 2 3 4 5 6 8 9 10 12 15 16 18 20 24 25 27 30 32 36 40 45 48 50 54 60 64 72 75 80 81 90 96 100 …

Call two numbers in this sequence adjacent, if the two numbers differ only by 1. The ten pairs (1,2), (2,3), (3,4), (4,5), (5,6), (8,9), (9,10), (15,16), (24,25), (80,81) correspond to the basic musical intervals from octave to diatonic comma. By computer, we have developed the sequence up to 10^{12}. No further pair of adjacent numbers has been found. The problem is to prove, that indeed (80, 81) is the last pair of adjacent numbers.

(7) The sound of a mathematician

A totally different interpretation of this paper's title is as follows: since mathematics is produced by mere mortal brains, we can try and listen to an encephalographic signal, which is a recording of the brain waves from both hemispheres of a mathematician. The signal has been recorded during a REM-phase and we hope that our mathematician was subconsciously doing some great mathematics while dreaming.

We transposed the signal through 7 octaves into the audible range and applied a few natural transformations in order to produce an eight channel "raumklang". It has been used as sound of a firestorm in recent stage performances of Henrik Ibsens' "Gespenster" at the Schauspielhaus in Düsseldorf. In a sense, however, this is the true sound of mathematics.